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Heat conduction between confocal elliptical surfaces using the theory of a Cosserat shell

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Abstract

This paper is concerned with steady-state heat conduction in rigid shell-like interphase regions. By analogy this work may provide insight into related problems of electric, dielectric and magnetic behavior. Although the field equations for three-dimensional linear Fourier heat condition are rather simple, the solution of problems in shell regions is significantly complicated when the shell has a general geometry and variable thickness. Here, the problem of heat conduction between confocal elliptical surfaces is solved within the context of the theory of a Cosserat shell. This problem is of particular interest because the Cosserat solution can be compared with an exact solution and the influences of variable shell thickness and strong variations of the temperature field through the shell's thickness can be explored independently. The results show that the Cosserat approach is reasonably accurate even for moderately thick shells, moderate ellipticity, and moderately strong variation of the temperature through the shell's thickness.

Keywords: Confocal ellipses; Cosserat shell; Heat conduction

1. Introduction

The response of composite materials to mechanical and thermal loads requires the solution of field equations in each material region as well as boundary conditions at common interfaces and at the outer boundaries of the composite. Sometimes the composite has components like particles and fibers which are covered by thin coatings. Other times thin regions occur near common boundaries which have different properties than the neighboring media either due to damage or chemical reactions caused by processes like gluing. Such thin regions are called interphases (Hashin, 2001, 2002).

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Nomenclature

$\mathbf{a}_\alpha$	tangent vectors to the shell's reference surface $\mathbf{x}$
$\mathbf{a}_3$	unit normal vector to the shell's reference surface $\mathbf{x}$
H	variable thickness of the shell
k	heat conduction coefficient
$\bar{\mathbf{n}}^*$	unit outward normal to the shell's bottom surface
$\hat{\mathbf{n}}^*$	unit outward normal to the shell's top surface
q^i, $q^{3\alpha}$	heat fluxes in the shell which require constitutive equations
$\bar{q}$	normal component of the heat flux into the shell through its bottom surface
$\hat{q}$	normal component of the heat flux out of the shell through its top surface
R	variable radius of curvature of the shell
$\mathbf{x}$	position vector to points on the shell's reference surface
$\mathbf{x}^*$	position vector to points in the three-dimensional shell region
$\bar{\mathbf{x}}$	position vector to points on the shell's bottom surface
$\hat{\mathbf{x}}$	position vector to points on the shell's top surface
$\bar{\alpha}$	scalar related to the area of the shell's bottom surface
$\hat{\alpha}$	scalar related to the area of the shell's top surface
$\kappa = H/R$	normalized thickness of the shell
θ^α	convected coordinates characterizing points on the shell's reference surface
θ^3	convected coordinate through the shell's thickness
θ	average temperature in the shell
θ^*	three-dimensional temperature field in the shell
θ_3	average temperature gradient (with respect to the thickness coordinate θ^3) through the shell's thickness
$\bar{\theta}$	temperature on the shell's bottom surface
$\hat{\theta}$	temperature on the shell's top surface

Most previous research has focused on replacing these interphases with imperfect interface conditions which require continuity or specify values of jumps in relevant quantities in the neighboring media to the interphase. Within the context of heat conduction, reference is made to the pioneering works of Sanchez-Palencia (1970) and Pham Huy and Sanchez-Palencia (1974) and to the more recent discussions by Benveniste (1987), Benveniste and Miloh (1986) and Miloh and Benveniste (1999), where a number of references can be found. Within the context of elasticity theory mention is made of the work of Benveniste and Miloh (2001) which provides an asymptotic analysis of various imperfect interface conditions.

The work of Hashin (2001), related to heat conduction, and Hashin (2002), related to elasticity, differ from previous work in that the response of the interphase is modeled using approximate shell-type equations for thin interphases. Within the context of elasticity, Benveniste and Miloh (2001) also showed connections with specific shell-type approximations of the interphase region. More recently, Rubin and Benvenste (2004) have used the theory of a Cosserat shell (Naghdi, 1972; Rubin, 2000) to model the response of an elastic interphase. In this work, the Cosserat model treats the interphase as having finite thickness and the boundary conditions are applied to the major surfaces of the shell. This is in contrast with previous work which attempts to develop imperfect interface conditions applied to a single infinitesimally thin interface surface. It was shown in Rubin and Benvenste (2004) that the Cosserat approach produces excellent results for elasticity even for moderately thick interphases.

This paper is concerned with heat conduction in rigid shell-like interphase regions. By analogy this work may provide insight into related problems of electric, dielectric and magnetic behavior. Although the field equations for three-dimensional linear Fourier heat condition are rather simple, the solution of problems in shell regions is significantly complicated when the shell has a general geometry and variable thickness. Consequently, it is desirable to develop simplified approximate equations to characterize heat conduction in general shells. Such equations have recently been developed (Rubin, 2004) where specific attention has been focused on developing constitutive equations which satisfy restrictions that ensure consistency with exact solutions for all constant temperature gradients and all shell geometries including variable thickness. In Rubin (2004) problems for circular cylindrical shells and spherical shells with constant thicknesses and constant curvatures were considered. It was shown that two other theories from the literature predict solutions which do not have the correct slope in the limit that the shell becomes vanishingly thin. In contrast, the Cosserat theory predicts solutions which converge smoothly to the exact solution in this limit. Moreover, it was shown that the Cosserat theory predicted reasonably accurate results relative to exact solutions even for moderately thick shells with moderately strong temperature variations through the thickness.

Here, the problem of heat conduction between confocal elliptical surfaces is solved within the context of the theory of a Cosserat shell. This problem is of particular interest because the Cosserat solution can be compared with an exact solution and the influences of variable shell thickness and strong variations of the temperature field through the shell's thickness can be explored independently. The results show that the Cosserat approach is quite accurate even for moderately thick shells, moderate ellipticity, and moderately strong variation of the temperature through the shell's thickness. Also, mention is made of the recent work by Chen (2004) on a confocally multicoated elliptical inclusion where exact solutions for antiplane shear (analogous to heat conduction) have been obtained.

Section 2 summarizes the equations for steady-state heat conduction in a rigid Cosserat shell which were developed in Rubin (2004). Section 3 presents the exact solution between confocal elliptical surfaces using the formulation in Carslaw and Jaeger (1956). Specifically, the exact solution for general boundary conditions is expressed in terms of a Fourier-type series. Section 4 develops the Cosserat solution for a typical term in this series so that the results can be used to determine the accuracy of the Cosserat equations for the entire exact solution. Finally, Section 5 presents a discussion and considers specific example problems which indicate that the Cosserat theory predicts reasonably accurate results even for moderately thick shells and moderately strong temperature variations through the shell's thickness.

2. Steady-state heat conduction in a rigid Cosserat shell

Within the context of the direct approach to the Cosserat theory of rigid heat conducting shells, the shell is described by the position vector $\mathbf{x}$ to points on its middle surface and by its thickness H

$$\{\mathbf{x}(\theta^\alpha), H(\theta^\alpha)\}, \tag{1}$$

which are both functions of two convected coordinates θ^α. Throughout the text, indices denoted by Greek letters take the values $(1,2)$, indices denoted by Latin letters take the values $(1,2,3)$ and the usual summation convention over the range of the index is used for repeated indices, Also, the temperature field is characterized by the average temperature θ and average temperature gradient θ_3 through the thickness of the shell

$$\{\theta(\theta^\alpha), \theta_3(\theta^\alpha)\}. \tag{2}$$

Alternatively, within the context of the three-dimensional approach a material point in the shell is characterized by the position vector $\mathbf{x}^*$ and the temperature at that point is characterized by θ^*, such that

$$\mathbf{x}^*(\theta^i) = \mathbf{x} + \theta^3 \mathbf{a}_3, \quad \theta^*(\theta^i) = \theta + \theta^3 \theta_3, \quad -\frac{H}{2} \leqslant \theta^3 \leqslant \frac{H}{2}, \tag{3a, b, c}$$

where $\mathbf{a}_3$ is the unit normal vector to the surface $\mathbf{x}$, and the convected coordinate θ^3 through the thickness of the shell should not be confused with the average temperature gradient θ_3. Here, and throughout the text a superposed (*) is used to denote quantities related to the exact three-dimensional solution which have the same symbol in the Cosserat theory. Also, it is convenient to define the position vectors $\{\bar{\mathbf{x}}, \hat{\mathbf{x}}\}$ to material points on the bottom ($\theta^3 = -H/2$) and top ($\theta^3 = H/2$) surfaces of the shell, respectively. Furthermore, it is convenient to define the temperatures $\{\bar{\theta}, \hat{\theta}\}$ on the shell's bottom and top surfaces, respectively, such that

$$\bar{\mathbf{x}}(\theta^\alpha) = \mathbf{x} - \frac{H}{2}\mathbf{a}_3, \quad \hat{\mathbf{x}} = \mathbf{x} + \frac{H}{2}\mathbf{a}_3, \quad \bar{\theta} = \theta - \frac{H}{2}\theta_3, \quad \hat{\theta} = \theta + \frac{H}{2}\theta_3. \tag{4}$$

It then follows that the Cosserat quantities can be determined by the surface variables through the equations

$$\mathbf{x} = \frac{1}{2}(\bar{\mathbf{x}} + \hat{\mathbf{x}}), \quad H = \mathbf{a}_3 \cdot (\hat{\mathbf{x}} - \bar{\mathbf{x}}), \quad \theta = \frac{1}{2}(\bar{\theta} + \hat{\theta}), \quad \theta_3 = \frac{1}{H}(\hat{\theta} - \bar{\theta}). \tag{5}$$

Moreover, it was shown in Rubin (2004) that the steady-state equations for heat conduction can be written in the forms:

$$\bar{q} = \frac{1}{\bar{\alpha}}\left[\frac{1}{2}q^\alpha_{,\alpha} + \frac{1}{H}(q^3 - q^{3\alpha}_{,\alpha})\right], \quad \hat{q} = -\frac{1}{\hat{\alpha}}\left[\frac{1}{2}q^\alpha_{,\alpha} - \frac{1}{H}(q^3 - q^{3\alpha}_{,\alpha})\right], \tag{6}$$

where $\bar{q}$ is the normal component of the heat flux into the shell through its bottom surface, $\hat{q}$ is the normal component of the heat flux out of the shell through its top surface, the heat fluxes

$$\{q^i, q^{3\alpha}\} \tag{7}$$

require constitutive equations which depend on the shell geometry (1) and depend linearly on the temperature fields

$$\{\theta_{,\alpha}, \theta_3, \theta_{3,\alpha}\}, \tag{8}$$

where a comma denotes partial differentiation with respect to coordinates θ^α. The tangent vectors $\mathbf{a}_\alpha$ and the normal $\mathbf{a}_3$ to the shell's middle surface, the scalar $a^{1/2}$, the reciprocal vectors $\mathbf{a}^\alpha$, and the metric $a^{\alpha\beta}$ are defined by

$$\begin{aligned} &\mathbf{a}_\alpha = \mathbf{x}_{,\alpha}, \quad \mathbf{a}_3 = a^{-1/2}(\mathbf{a}_1 \times \mathbf{a}_2), \quad a^{1/2} = |\mathbf{a}_1 \times \mathbf{a}_2|, \\ &\mathbf{a}^1 = a^{-1/2}(\mathbf{a}_2 \times \mathbf{a}_3), \quad \mathbf{a}^2 = a^{-1/2}(\mathbf{a}_3 \times \mathbf{a}_1), \quad a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta. \end{aligned} \tag{9}$$

In addition, the quantities $\{\bar{\alpha}, \hat{\alpha}\}$ in (6) and the unit outward normal vectors $\{\bar{\mathbf{n}}^*, \hat{\mathbf{n}}^*\}$ are related to the shell's bottom and top surfaces, respectively, and are defined by

$$\begin{aligned} &\bar{\alpha}\bar{\mathbf{n}}^* = -\left(\mathbf{x} - \frac{H}{2}\mathbf{a}_3\right)_{,1} \times \left(\mathbf{x} - \frac{H}{2}\mathbf{a}_3\right)_{,2}, \quad \bar{\mathbf{n}}^* \cdot \bar{\mathbf{n}}^* = 1, \\ &\hat{\alpha}\hat{\mathbf{n}}^* = \left(\mathbf{x} + \frac{H}{2}\mathbf{a}_3\right)_{,1} \times \left(\mathbf{x} + \frac{H}{2}\mathbf{a}_3\right)_{,2}, \quad \hat{\mathbf{n}}^* \cdot \hat{\mathbf{n}}^* = 1. \end{aligned} \tag{10}$$

Also, the magnitude of the thickness H is limited by the condition that (3a) characterizes a one-to-one mapping between the convected coordinates θ^i and material points in the shell. Specifically, H is limited by the condition that

$$(\mathbf{a}_1 + \theta^3 \mathbf{a}_{3,1}) \times (\mathbf{a}_2 + \theta^3 \mathbf{a}_{3,2}) \cdot \mathbf{a}_3 \geqslant 0 \tag{11}$$

which is required for all values of θ^i.

In Rubin (2004) restrictions on the constitutive equations for (7) were developed which ensure that Eqs. (6) are consistent with the exact three-dimensional solution for an arbitrary constant three-dimensional temperature gradient and arbitrary shell geometry including variable thickness. For Fourier heat conduction, the constitutive equations for $\{q^\alpha, q^{3\alpha}\}$ were specified by generalizations of Bubnov–Galerkin forms for a plate

$$q^\alpha = -kH[a^{1/2}a^{\alpha\beta}]\theta_{,\beta}, \quad q^{3\alpha} = -\frac{kH^3}{12}[a^{1/2}a^{\alpha\beta}]\theta_{3,\beta} \tag{12}$$

and the constitutive equation for $\{q^3\}$ was modified so that the complete set of constitutive equations satisfies these restrictions for shells

$$\begin{aligned} q^3 &= -kH[B^\alpha\theta_{,\alpha} + B^3\theta_3], \\ B^\alpha &= \left[\frac{H^2}{12}(a^{1/2}a^{\sigma\gamma}\mathbf{a}_{3,\sigma})_{,\gamma} + \frac{H}{4}\{H_{,\sigma}a^{1/2}a^{\sigma\gamma}\mathbf{a}_{3,\gamma} - H_{,1}(\mathbf{a}_{3,2}\times\mathbf{a}_3) - H_{,2}(\mathbf{a}_3\times\mathbf{a}_{3,1})\}\right]\cdot\mathbf{a}^\alpha, \\ B^3 &= \left[a^{1/2} + \frac{H^2}{4}(\mathbf{a}_{3,1}\times\mathbf{a}_{3,2}\cdot\mathbf{a}_3) + \frac{H^2}{12}\{(a^{1/2}a^{\sigma\gamma}\mathbf{a}_{3,\sigma})_{,\gamma}\cdot\mathbf{a}_3\}\right], \end{aligned} \tag{13}$$

where k is the constant heat conduction coefficient. In particular, it can be seen that these restrictions are nontrivial since q^3 requires a rather complicated dependence on the shell geometry. Further, in this regard, it was shown in Rubin (2004) that these restrictions are not satisfied by three other models (Rubin, 1986; Lukasiewicz, 1989; Hashin, 2001) for heat conduction in rigid shells.

In summary, Eqs. (6) connect the heat fluxes through the shell's bottom and top surfaces to derivatives of the shell's geometry $\{\bar{\mathbf{x}}, \hat{\mathbf{x}}\}$ and the temperature fields $\{\bar{\theta}, \hat{\theta}\}$ on those surfaces through the constitutive equations (12) and (13).

3. Heat conduction between confocal elliptical surfaces: exact solution

The accuracy of the Cosserat theory in Section 2 was examined in Rubin (2004) by considering a number of example problems which included heat conduction in plates, circular cylindrical and spherical regions. However, all of these examples considered shells with constant curvature and constant thickness. In this section the accuracy of the Cosserat theory is examined by considering the example of two-dimensional heat conduction between two confocal ellipses. This example represents a severe test of the theory because both the curvature and thickness of the shell are variable and extreme limiting cases can be considered and compared with an exact solution.

Following (Carslaw and Jaeger, 1956, pp. 439–440) the position vector in a shell-like elliptical region can be characterized by

$$\mathbf{x}^*(\theta^i) = \mathbf{x}^*(\alpha, z, \beta) = c\cosh(\beta)\cos\alpha\mathbf{e}_1 + c\sinh(\beta)\sin\alpha\mathbf{e}_2 + z\mathbf{e}_3, \quad \theta^1 = \alpha, \quad \theta^2 = z, \quad \theta^3 = \beta. \tag{14}$$

Moreover, the boundaries of the elliptical regions are specified by

$$0 \leqslant \alpha \leqslant 2\pi, \quad -\infty \leqslant z \leqslant \infty, \quad \beta_1 = \gamma - \delta \leqslant \beta \leqslant \beta_2 = \gamma + \delta, \tag{15}$$

where c, γ and δ are positive constants. For steady-state Fourier heat conduction the heat flux vector $\mathbf{q}^*$ satisfies the equations

$$\mathbf{q}^* = -k\mathbf{g}^*, \quad \mathbf{g}^* = \frac{\partial\theta^*}{\partial\mathbf{x}^*}, \quad \mathrm{div}^*\mathbf{q}^* = 0, \tag{16a, b, c}$$

where $\mathbf{g}^*$ is the three-dimensional temperature gradient and div* is the divergence operator with respect to $\mathbf{x}^*$.

In order to write expressions for the gradient and divergence operators in these elliptical coordinates it is convenient to use Appendix A in Rubin (2000) to introduce the covariant base vectors $\mathbf{g}_i$, the scalar $g^{1/2}$ and the reciprocal vectors $\mathbf{g}^i$ by the formulas

$$\begin{aligned}
&\mathbf{g}_1 = \mathbf{x}^*_{,1} = -c\cosh(\beta)\sin\alpha\mathbf{e}_1 + c\sinh(\beta)\cos\alpha\mathbf{e}_2, \quad \mathbf{g}_2 = \mathbf{x}^*_{,2} = \mathbf{e}_2,\\
&\mathbf{g}_3 = \mathbf{x}^*_{,3} = c\sinh(\beta)\cos\alpha\mathbf{e}_1 + c\cosh(\beta)\sin\alpha\mathbf{e}_2,\\
&g^{1/2} = \mathbf{g}_1\times\mathbf{g}_2\cdot\mathbf{g}_3 = c^2[\sinh^2(\beta)\cos^2\alpha + \cosh^2(\beta)\sin^2\alpha],\\
&g^{1/2}\mathbf{g}^1 = \mathbf{g}_2\times\mathbf{g}_3 = -c\cosh(\beta)\sin\alpha\mathbf{e}_1 + c\sinh(\beta)\cos\alpha\mathbf{e}_2,\\
&g^{1/2}\mathbf{g}^2 = \mathbf{g}_3\times\mathbf{g}_1 = c^2[\sinh^2(\beta)\cos^2\alpha + \cosh^2(\beta)\sin^2\alpha]\mathbf{e}_3,\\
&g^{1/2}\mathbf{g}^3 = \mathbf{g}_1\times\mathbf{g}_2 = c\sinh(\beta)\cos\alpha\mathbf{e}_1 + c\cosh(\beta)\sin\alpha\mathbf{e}_2.
\end{aligned} \tag{17}$$

Then, it can be shown that

$$\mathbf{g}^* = \theta^*_{,i}\mathbf{g}^i, \quad g^{1/2}\,\mathrm{div}^*\mathbf{q}^* = (g^{1/2}\mathbf{q}^*\cdot\mathbf{g}^i)_{,i}. \tag{18}$$

In particular, it follows that for the two-dimensional problem under consideration

$$\begin{aligned}
&\theta^* = \theta^*(\alpha,\beta),\\
&\mathbf{q}^* = -c\mathrm{k}g^{-1/2}\left[\frac{\partial\theta^*}{\partial\alpha}\{-\cosh(\beta)\sin\alpha\mathbf{e}_1 + \sinh(\beta)\cos\alpha\mathbf{e}_2\} + \frac{\partial\theta^*}{\partial\beta}\{\sinh(\beta)\cos\alpha\mathbf{e}_1 + \cosh(\beta)\sin\alpha\mathbf{e}_2\}\right],\\
&g^{1/2}\,\mathrm{div}^*\mathbf{q}^* = -k\left[\frac{\partial^2\theta^*}{\partial\alpha^2} + \frac{\partial^2\theta^*}{\partial\beta^2}\right] = 0.
\end{aligned} \tag{19}$$

Also, the values of $\{\bar{\alpha},\hat{\alpha}\}$ and the unit outward normals $\{\bar{\mathbf{n}}^*,\hat{\mathbf{n}}^*\}$ to the shell's bottom and top surfaces [which are consistent with (10)] are given by

$$\begin{aligned}
&\bar{\alpha}\bar{\mathbf{n}}^* = -\mathbf{x}^*(\alpha,z,\beta_1)_{,1}\times\mathbf{x}^*(\alpha,z,\beta_1)_{,2} = -c[\sinh(\beta_1)\cos\alpha\mathbf{e}_1 + \cosh(\beta_1)\sin\alpha\mathbf{e}_2],\\
&\bar{\alpha} = c\sqrt{\sinh^2(\beta_1)\cos^2\alpha + \cosh^2(\beta_1)\sin^2\alpha},\\
&\hat{\alpha}\hat{\mathbf{n}}^* = \mathbf{x}^*(\alpha,z,\beta_2)_{,1}\times\mathbf{x}^*(\alpha,z,\beta_2)_{,2} = c[\sinh(\beta_2)\cos\alpha\mathbf{e}_1 + \cosh(\beta_2)\sin\alpha\mathbf{e}_2],\\
&\hat{\alpha} = c\sqrt{\sinh^2(\beta_2)\cos^2\alpha + \cosh^2(\beta_2)\sin^2\alpha}
\end{aligned} \tag{20}$$

so that the normal component $\bar{q}^*$ of heat flux into the shell through its bottom surface, and the normal component $\hat{q}^*$ of the heat flux out of the shell through its top surface become

$$\begin{aligned}
&\bar{\alpha}\bar{q}^* = -\mathbf{q}^*(\alpha,\beta_1)\cdot\bar{\alpha}\bar{\mathbf{n}}^* = -k\frac{\partial\theta^*}{\partial\beta}(\alpha,\beta_1),\\
&\hat{\alpha}\hat{q}^* = \mathbf{q}^*(\alpha,\beta_2)\cdot\hat{\alpha}\hat{\mathbf{n}}^* = -k\frac{\partial\theta^*}{\partial\beta}(\alpha,\beta_2).
\end{aligned} \tag{21}$$

It was shown in Carslaw and Jaeger (1956, pp. 439–440) that an exact solution can be obtained in terms of a Fourier series for any distribution of temperature on surfaces of the shell. Here, typical cases of this exact solution are considered by taking

$$\theta^* = \theta_0 + \overline{\Theta}^* \left[\frac{\sinh\{m(\gamma+\delta-\beta)\}}{\sinh(2m\delta)} \right] \cos(m\alpha) + \hat{\Theta}^* \left[\frac{\sinh\{n(\beta-\gamma+\delta)\}}{\sinh(2n\delta)} \right] \cos(n\alpha),$$
$$\bar{\theta}(\alpha) = \theta^*(\alpha, \beta_1) = \theta_0 + \overline{\Theta}^* \cos(m\alpha), \quad \hat{\theta}(\alpha) = \theta^*(\alpha, \beta_2) = \theta_0 + \widehat{\Theta}^* \cos(n\alpha), \tag{22}$$
$$\frac{\sinh\{m(\gamma+\delta-\beta)\}}{\sinh(2m\delta)} = \frac{\gamma+\delta-\beta}{2\delta} \quad \text{for } m=0, \quad \frac{\sinh\{n(\beta-\gamma+\delta)\}}{\sinh(2n\delta)} = \frac{\beta-\gamma+\delta}{2\delta} \quad \text{for } n=0,$$

where θ_0 is the constant reference temperature, m and n are integers, and the constants $\{\overline{\Theta}^*, \widehat{\Theta}^*\}$ are the Fourier coefficients of the temperature fields on the shell's bottom and top surfaces, respectively. It then follows that the exact heat fluxes can be expressed in the forms:

$$\bar{\alpha}\bar{q}^* = k[\overline{M}^*\overline{\Theta}^* \cos(m\alpha) - \overline{N}^* \widehat{\Theta}^* \cos(n\alpha)],$$
$$\hat{\alpha}\hat{q}^* = k[\widehat{M}^*\overline{\Theta}^* \cos(m\alpha) - \widehat{N}^* \widehat{\Theta}^* \cos(n\alpha)], \tag{23}$$

where the constants $\{\overline{M}^*, \overline{N}^*, \widehat{M}^*, \widehat{N}^*\}$ are given by

$$\overline{M}^* = \frac{m}{\tanh(2m\delta)}, \quad \overline{N}^* = \frac{n}{\sinh(2n\delta)}, \quad \widehat{M}^* = \frac{m}{\sinh(2m\delta)}, \quad \widehat{N}^* = \frac{n}{\tanh(2n\delta)},$$
$$\frac{m}{\tanh(2m\delta)} = \frac{1}{2\delta} \quad \text{for } m=0, \quad \frac{n}{\sinh(2n\delta)} = \frac{1}{2\delta} \quad \text{for } n=0. \tag{24}$$

4. Heat conduction between confocal elliptical surfaces: Cosserat solution

Using (14) it follows that the position vectors $\{\bar{\mathbf{x}}, \hat{\mathbf{x}}\}$ are given by

$$\bar{\mathbf{x}}(\alpha, z) = \mathbf{x}^*(\alpha, \beta_1, z), \quad \hat{\mathbf{x}}(\alpha, z) = \mathbf{x}^*(\alpha, \beta_2, z) \tag{25}$$

so that with the help of (5) the middle surface of the Cosserat shell can be taken in the form:

$$\mathbf{x} = A\cos\alpha\mathbf{e}_1 + B\sin\alpha\mathbf{e}_2 + z\mathbf{e}_3, \quad \theta^1 = \alpha, \quad \theta^2 = z, \tag{26}$$

where the constant c in (14) and the constants A, B associated with the major and minor axes of the ellipse, respectively, are defined so that

$$c = A\,\mathrm{sech}(\gamma)\,\mathrm{sech}(\delta), \quad \frac{B}{A} = \tanh(\gamma). \tag{27a, b}$$

Thus, the value of γ determines the ellipticity ratio B/A. Now, with the help of (9) the kinematics of the shell become

$$\mathbf{a}_1 = -A\sin\alpha\mathbf{e}_1 + B\cos\alpha\mathbf{e}_2, \quad \mathbf{a}_2 = \mathbf{e}_3,$$
$$\mathbf{a}_3 = a^{-1/2}(B\cos\alpha\mathbf{e}_1 + A\sin\alpha\mathbf{e}_2), \quad a^{1/2} = \sqrt{A^2\sin^2\alpha + B^2\cos^2\alpha}, \tag{28}$$
$$\mathbf{a}^1 = a^{-1}(-A\sin\alpha\mathbf{e}_1 + B\cos\alpha\mathbf{e}_2), \quad \mathbf{a}^2 = \mathbf{e}_3, \quad a^{11} = a^{-1}, \quad a^{12} = 0, \quad a^{22} = 1.$$

Also, using (5) it follows that the shell's thickness H is given by

$$H = 2\tanh(\delta)a^{1/2}. \tag{29}$$

Next, with the help of (5) and (22) the Cosserat temperature fields become

$$\theta = \theta(\alpha) = \theta_0 + \frac{1}{2}(\overline{\Theta} + \widehat{\Theta}), \quad \theta_3 = \theta_3(\alpha) = \frac{1}{H}(\widehat{\Theta} - \overline{\Theta}). \tag{30}$$

Thus, using the results

$$a_{,1} = (A^2 - B^2)\sin(2\alpha), \quad H_{,1} = \frac{1}{2}(A^2 - B^2)a^{-1}H\sin(2\alpha) \tag{31}$$

it follows that

$$\theta_{,1} = \frac{1}{2}\left[\frac{\partial\overline{\Theta}}{\partial\alpha} + \frac{\partial\widehat{\Theta}}{\partial\alpha}\right], \quad \theta_{3,1} = \frac{1}{H}\left[\frac{\partial\widehat{\Theta}}{\partial\alpha} - \frac{\partial\overline{\Theta}}{\partial\alpha}\right] - \left[\frac{1}{2}(A^2 - B^2)a^{-1}\sin(2\alpha)\right]\frac{1}{H}[\widehat{\Theta} - \overline{\Theta}]. \tag{32}$$

Consequently, with the help of (12), (29) and (32) it can be shown that

$$\begin{aligned}
q^{\alpha}_{,\alpha} &= -k[\tanh(\delta)]\left[\frac{\partial^2\overline{\Theta}}{\partial\alpha^2} + \frac{\partial^2\widehat{\Theta}}{\partial\alpha^2}\right],\\
q^{3\alpha}_{,\alpha} &= -\frac{k}{3}[\tanh^2(\delta)]a^{1/2}\left[\frac{\partial^2\widehat{\Theta}}{\partial\alpha^2} - \frac{\partial^2\overline{\Theta}}{\partial\alpha^2}\right]\\
&\quad - \frac{k}{12}[(A^2 - B^2)\tanh^2(\delta)][(A^2 - B^2)\sin^2(2\alpha) - 4a\cos(2\alpha)]a^{-3/2}[\widehat{\Theta} - \overline{\Theta}].
\end{aligned} \tag{33}$$

Moreover, using (28) it follows that

$$\begin{aligned}
&\mathbf{a}_{3,1} = a^{-1/2}(-B\sin\alpha\mathbf{e}_1 + A\cos\alpha\mathbf{e}_2) - \frac{1}{2}a^{-3/2}(A^2 - B^2)\sin(2\alpha)(B\cos\alpha\mathbf{e}_1 + A\sin\alpha\mathbf{e}_2),\\
&\mathbf{a}_{3,1}\cdot\mathbf{a}^1 = ABa^{-3/2},\\
&(a^{1/2}a^{\sigma\gamma}\mathbf{a}_{3,\sigma})_{,\gamma} = \Big[-\frac{3}{2}a^{-2}(A^2 - B^2)\sin(2\alpha)(-B\sin\alpha\mathbf{e}_1 + A\cos\alpha\mathbf{e}_2)\\
&\qquad + \{a^{-3}(A^2 - B^2)^2\sin^2(2\alpha) - a^{-2}(A^2 - B^2)\cos(2\alpha) - a^{-1}\}(B\cos\alpha\mathbf{e}_1 + A\sin\alpha\mathbf{e}_2)\Big],\\
&(a^{1/2}a^{\sigma\gamma}\mathbf{a}_{3,\sigma})_{,\gamma}\cdot\mathbf{a}^1 = -\frac{3}{2}a^{-3}AB(A^2 - B^2)\sin(2\alpha),\\
&(a^{1/2}a^{\sigma\gamma}\mathbf{a}_{3,\sigma})_{,\gamma}\cdot\mathbf{a}_3 = \left[\frac{1}{4}a^{-5/2}(A^2 - B^2)^2\sin^2(2\alpha) - a^{-3/2}(A^2 - B^2)\cos(2\alpha) - a^{-1/2}\right].
\end{aligned} \tag{34}$$

Thus, (13) and (30) yield

$$\begin{aligned}
B^{\alpha} &= 0,\\
q^3 &= -k\left[1 - \frac{\tanh^2(\delta)}{3}\right]a^{1/2}(\widehat{\Theta} - \overline{\Theta})\\
&\quad - \frac{k}{12}[(A^2 - B^2)\tanh^2(\delta)][(A^2 - B^2)\sin^2(2\alpha) - 4a\cos(2\alpha)]a^{-3/2}(\widehat{\Theta} - \overline{\Theta}).
\end{aligned} \tag{35}$$

It then follows that the Cosserat values of the heat fluxes (6) are given by

$$\begin{aligned}
\bar{\alpha}\bar{q} &= -k\left[\frac{1}{2\tanh(\delta)}\right]\left[1 - \frac{\tanh^2(\delta)}{3}\right](\widehat{\Theta} - \overline{\Theta}) - \frac{2k}{3}[\tanh(\delta)]\frac{\partial^2\overline{\Theta}}{\partial\alpha^2} - \frac{k}{3}[\tanh(\delta)]\frac{\partial^2\widehat{\Theta}}{\partial\alpha^2},\\
\hat{\alpha}\hat{q} &= -k\left[\frac{1}{2\tanh(\delta)}\right]\left[1 - \frac{\tanh^2(\delta)}{3}\right](\widehat{\Theta} - \overline{\Theta}) + \frac{k}{3}[\tanh(\delta)]\frac{\partial^2\overline{\Theta}}{\partial\alpha^2} + \frac{2k}{3}[\tanh(\delta)]\frac{\partial^2\widehat{\Theta}}{\partial\alpha^2},
\end{aligned} \tag{36}$$

where the values of $\{\bar{\alpha}, \hat{\alpha}\}$ in (10) are the same as those in (20) associated with the exact solution, and can be obtained by using the definitions of $\{\beta_1, \beta_2\}$ in (15) and c in (27).

Also, with the help of (11), (28), (29), (34) it can be shown that

$$(\mathbf{a}_1 + \theta^3\mathbf{a}_{3,1}) \times (\mathbf{a}_2 + \theta^3\mathbf{a}_{3,2}) \cdot \mathbf{a}_3 = a^{1/2}[1 + \theta^3 a^{-3/2}AB] \geqslant 0 \tag{37}$$

so that for $\theta^3 = -H/2$ the value of δ is limited by the condition

$$0 \leqslant \tanh(\delta) \leqslant \frac{B}{A} \leqslant \frac{a}{AB} \quad \text{for } B \leqslant A, \quad \delta \leqslant \gamma = \tanh^{-1}\left(\frac{B}{A}\right), \tag{38a, b}$$

where use has been made of (27b). In particular, notice from (14) and (15) that when the equality holds in (38b), the bottom surface ($\beta = \beta_1 = 0$) converges to the $\mathbf{e}_1$–$\mathbf{e}_3$ plane and the elliptical region is solid. Moreover, it can be shown that for the surface (26) the variable radius of curvature R is given by

$$\frac{1}{R} = -\mathbf{a}_3 \cdot (a^{-1/2}\mathbf{a}_1)_{,1} a^{-1/2}, \quad R(\alpha) = a^{1/2}. \tag{39}$$

Consequently, with the help of (28) and (29) the normalized thicknesses can be defined so that

$$\kappa = \frac{H(\alpha)}{R(\alpha)} = 2\tanh(\delta) \leqslant \frac{2B}{A} \quad \text{for } B \leqslant A, \tag{40}$$

which is independent of the variable α.

For the special case when the temperature fields are specified by $\bar{\theta}$ and $\hat{\theta}$ in (22) it follows from (30) that

$$\overline{\Theta} = \overline{\Theta}^* \cos(m\alpha), \quad \widehat{\Theta} = \widehat{\Theta}^* \cos(n\alpha), \quad \text{for } m \geqslant 0 \text{ and } n \geqslant 0 \tag{41}$$

so that the heat fluxes (36) can be written in the forms:

$$\begin{aligned} \bar{\alpha}\bar{q}^* &= k[\overline{M}\,\overline{\Theta}^* \cos(m\alpha) - \overline{N}\widehat{\Theta}^* \cos(n\alpha)], \\ \hat{\alpha}\hat{q}^* &= k[\widehat{M}\overline{\Theta}^* \cos(m\alpha) - \widehat{N}\widehat{\Theta}^* \cos(n\alpha)], \end{aligned} \tag{42}$$

where the constants $\{\overline{M}, \overline{N}, \widehat{M}, \widehat{N}\}$ are given by

$$\begin{aligned} \overline{M} &= \left[\frac{1}{2\tanh(\delta)}\right]\left[1 + \frac{(4m^2-1)}{3}\tanh^2(\delta)\right], \quad \overline{N} = \left[\frac{1}{2\tanh(\delta)}\right]\left[1 - \frac{(2n^2+1)}{3}\tanh^2(\delta)\right], \\ \widehat{M} &= \left[\frac{1}{2\tanh(\delta)}\right]\left[1 - \frac{(2m^2+1)}{3}\tanh^2(\delta)\right], \quad \widehat{N} = \left[\frac{1}{2\tanh(\delta)}\right]\left[1 + \frac{(4n^2-1)}{3}\tanh^2(\delta)\right]. \end{aligned} \tag{43}$$

5. Discussion

Comparison of (23) with (42) indicates that the Cosserat solution preserves the character of the exact solution for all Fourier coefficients of the temperature fields on the shell's bottom and top surfaces. Only the magnitudes $\{\overline{M}, \overline{N}, \widehat{M}, \widehat{N}\}$ of the coefficients in the heat fluxes are approximate in the Cosserat solution. Moreover, it can be shown that the Cosserat values (43) are the same as the exact values (24) for $m = n = 1$ and for general elliptical shells with arbitrary shell thickness [δ or κ in (40)] and arbitrary ellipticity [γ or B/A in (27)]. This is a strong indication that the Cosserat theory can produce relatively accurate solutions for shells with variable curvature and variable thickness.

It is well known that the notion of a shell being "thin" is not purely geometric. It actually requires the variation of the important parameters through the shell's thickness to remain relatively small. In this regard, it is noted from (22) that the variation of the temperature field through the shell's thickness strengthens as the values of m and n increase. This means that the accuracy of the Cosserat theory can be tested by this example problem in both the limits that the shell becomes geometrically thicker (increasing κ) and in the limit that the temperature variation through its thickness becomes stronger (increasing values of m and n).

Since the structure of the solutions for the temperature fields $\bar{\theta}$ and $\hat{\theta}$ are similar it is sufficient to consider an example problem for which $\overline{\Theta}^*$ vanishes and the temperature remains constant on the shell's bottom surface. Moreover, it is convenient to use (24) and (43) to introduce the functions

$$\overline{E}(\delta, n) = \frac{\overline{N}}{\overline{N}^*} - 1, \quad \widehat{E}(\delta, n) = \frac{\widehat{N}}{\widehat{N}^*} - 1, \tag{44}$$

which measure the error in the Cosserat predictions relative to the exact solution. As previously mentioned these errors vanish for $n = 1$ for all confocal elliptical shells satisfying the restriction (40)

$$\overline{E}(\delta, 1) = \widehat{E}(\delta, 1) = 0. \tag{45}$$

Also, for $n = 0$ it can be shown that the errors remain very small for small values of δ since

$$\overline{E}(\delta, 0) = \widehat{E}(\delta, 0) = \delta\left[\coth(\delta) - \frac{1}{3}\tanh(\delta)\right] - 1 \approx \frac{4}{45}\delta^4 + O(\delta^6). \tag{46}$$

Fig. 1 shows the geometry of two shells with $\kappa = 0.2$ and two different values of ellipticity ($B/A = 0.5$ in Fig. 1a, and $B/A = 0.1$ in Fig. 1b). In particular, it can been seen that both the local curvature and the normal thickness vary significantly for these shells especially near the tips of the major axis of the ellipses. Fig. 2 plots the errors $\{\overline{E}, \widehat{E}\}$ in (44) versus the curvature κ for two values of the Fourier coefficients ($n = 5$ in Fig. 2a and $n = 10$ in Fig. 2b). Fig. 2c plots these errors versus the order n for the case of $\kappa = 0.1$. In particular, it can be seen that the magnitudes of the errors remain less than 2% for the case of $n = 5$ even for moderately thick shells with $\kappa = 0.2$. As expected, the error increases with increasing κ or increasing n. From (22) it can also be seen that the function

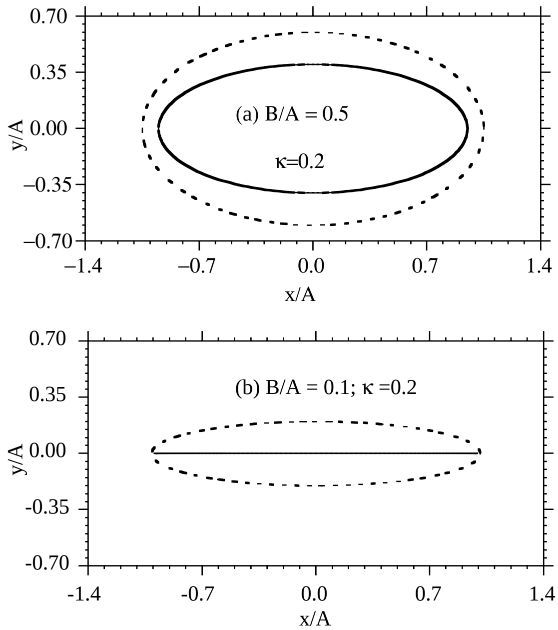

Fig. 1. Shapes of the shell for $\kappa = 0.2$ and different values of ellipticity B/A.

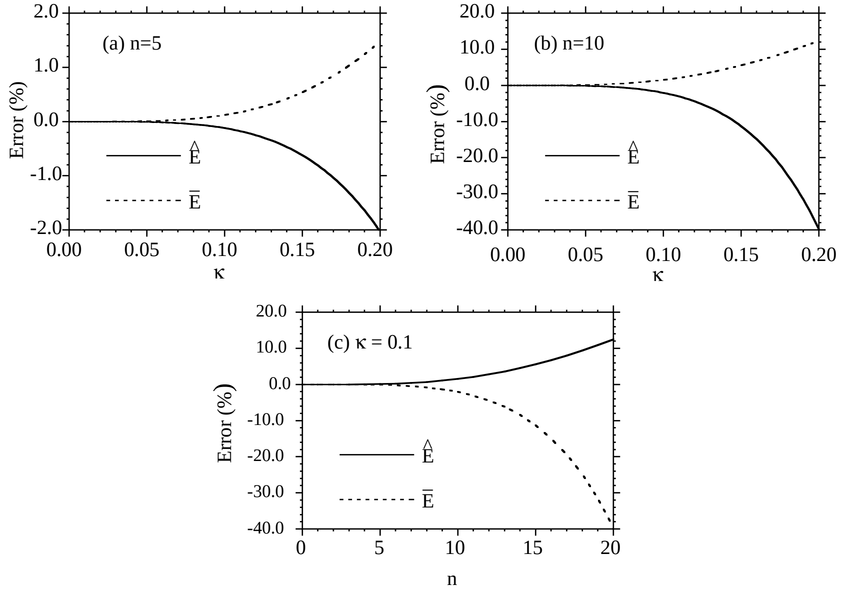

Fig. 2. Errors $\{\overline{E}, \widehat{E}\}$ in (44) as functions of the normalized thickness κ in (40) associated with the Cosserat solution for different values of n.

$$f(\beta) = \frac{\sinh\{n(\beta - \gamma + \delta)\}}{\sinh(2n\delta)} \tag{47}$$

characterizes the variation of the exact temperature field through the thickness of the shell region. Fig. 3 plots this function $f(\beta)$ for $\kappa = 0.2$ and different values of n and the ellipticity B/A [with γ given by (38b) and δ given by (40)].

In summary, the example of heat conduction between two confocal elliptical surfaces has been considered to test the Cosserat theory developed in Rubin (2004) for heat conduction in a rigid shell of arbitrary shape. Specifically, this example tests the Cosserat theory for a shell with both variable curvature and variable thickness. As expected, the error in the Cosserat theory increases as the shell becomes thicker and the variation of the temperature field through the shell's thickness strengthens. However, the results indicate that the Cosserat theory predicts reasonably accurate results even for moderately thick shells and moderately strong temperature variations through the shell's thickness.

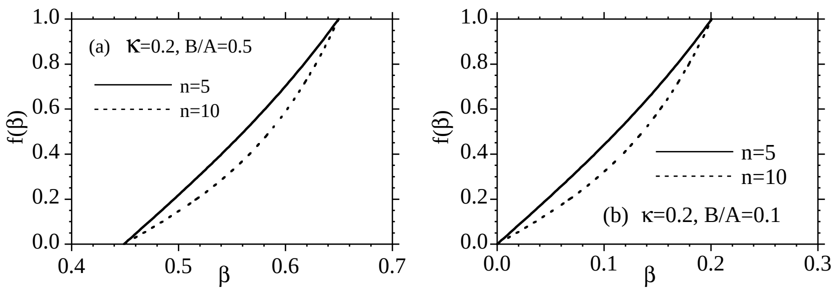

Fig. 3. Values of the function $f(\beta)$ in (47) characterizing the variation of temperature through the thickness of the shell for $\kappa = 0.2$ and different values of n and the ellipticity B/A.

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